Multiple Proofs for a Geometric Problem

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Multiple Proofs for a Geometric Problem

Introduction

The following is a typical plane geometry problem:

ABCD is a quadrilateral with $AB = CD$. Let *E* and *F* be the midpoints of *AD* and *BC* respectively. Lines *CD* and *FE* produced meet at point *G* and lines *BA* and *FE* produced meet at point *H*, as shown in Figure 1.
Prove that $\angle BHF = \angle CGF$.

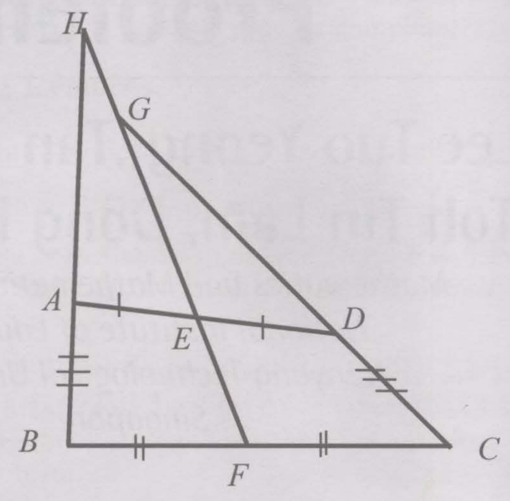

Figure 1

While the problem is not very difficult to solve, it will be instructive to elucidate from students as many different proofs as possible. This has the dual purpose of encouraging creativity and raising the important awareness that quite often, there are many ways (sometimes using different areas of mathematics) to solve a particular mathematics problem.

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To these ends, this paper presents ten different proofs of the problem, five complete and five sketches. These proofs involve the following areas in mathematics.

(i) Geometry
(ii) Trigonometry
(iii) Vectors
(iv) Analytic geometry

Constructing New Line Segments of Extending Old Ones

The method of constructing new line segments or extending old ones is very common in geometry proofs. One must not be afraid to 'dirty' the diagram with new lines and new points to get a better view of the problem. Line segments drawn (and their extensions) are usually parallel or perpendicular to existing lines or join existing points. Points are also suitably chosen (midpoints, points of intersection, etc.).

Proof 1

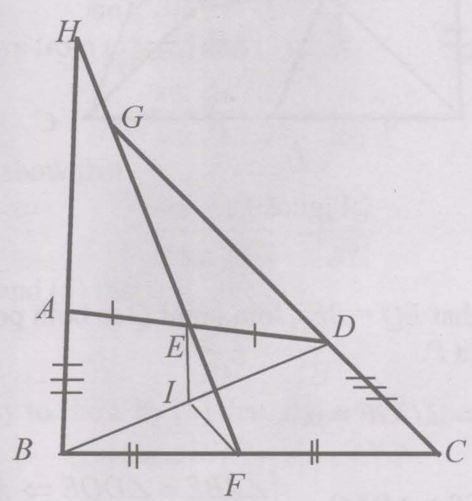

Figure 2

Join B and D, and let I be the mid-point of BD. Then join I to both E and F, as shown in Figure 2.

Since $AE = ED$ and $BI = ID$, we have $EI = \frac{1}{2} AB$ and $EI \parallel AB$. Similarly, we have $FI = \frac{1}{2} CD$ and $FI \parallel CD$.

Since $AB = CD$, we have $EI = FI$. Thus $\angle IEF = \angle IFE$. Since $EI \parallel AB$, we have $\angle IEF = \angle BHF$. Since $FI \parallel CD$, we have $\angle IFE = \angle CGF$.

Hence $\angle BHF = \angle CGF$.

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Note: Here we have used the fact that the line joining the mid-points of two sides of a triangle is always parallel to and half the length of the third side. This is a very useful fact in solving many geometry problems.

Proof 2

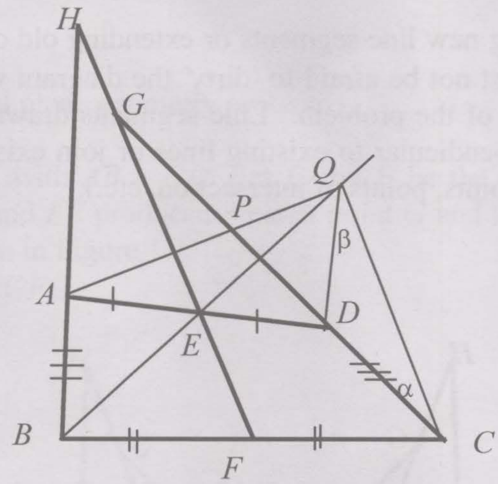

Figure 3

Extend BE to point Q such that $EQ = BE$. Join point Q to both points C and D. Let CG and AQ meet at point P.

$$\left.\begin{array}{l} BE = EQ \\ BF = FC \end{array}\right\} \Rightarrow EF \,//\, QC \Rightarrow \angle CGF = \alpha.$$

$$\left.\begin{array}{c} AE = DE \\ \angle AEB = \angle DEQ \\ BE = QE \end{array}\right\} \Rightarrow \Delta AEB \cong \Delta DEQ \Rightarrow \left\{\begin{array}{l} \angle ABE = \angle DQE \Rightarrow AB \,//\, DQ \\ AB = DQ. \end{array}\right.$$

$$\left.\begin{array}{l} AB = DQ \\ AB = DC \end{array}\right\} \Rightarrow DQ = DC \Rightarrow \alpha = \beta.$$

$$\left.\begin{array}{l} \angle BHF = 180^\circ - \angle BFH - \angle FBH \\ EF \,//\, CQ \Rightarrow \angle BFH = \angle BCQ \\ AB \,//\, QD \Rightarrow \angle HBE = \angle DQE \end{array}\right\} \Rightarrow \angle BHF = 180^\circ - \angle BCQ - \angle QBC - \angle DQE = \beta.$$

$$\left.\begin{array}{l} \alpha = \beta \\ \alpha = \angle CGF \\ \beta = \angle BHF \end{array}\right\} \Rightarrow \angle CGF = \angle BHF.$$

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Some Trigonometry – Using Sine Rule

The next proof involves the application of the sine rule.

Proof 3

Applying the sine rule to ΔBFH gives

$$\frac{\sin\angle BHF}{BF}=\frac{\sin\angle BFH}{BH}. \tag{1}$$

Applying the sine rule to ΔCFG gives

$$\frac{\sin\angle CGF}{FC}=\frac{\sin\angle CFG}{GC}. \tag{2}$$

Since $\angle BFH+\angle CFG=180°$, we have

$$\sin\angle BFH=\sin\angle CFG. \tag{3}$$

Since $BF=FC$, it follows from (1), (2) and (3) that

$$\frac{\sin\angle BHF}{\sin\angle CGF}=\frac{GC}{BH}. \tag{4}$$

In the same way, we can show that

$$\frac{\sin\angle BHF}{\sin\angle CGF}=\frac{DG}{AH}. \tag{5}$$

Thus it follows from (4) and (5) that

$$\frac{GC}{BH}=\frac{DG}{AH}. \tag{6}$$

Since $AB=DC$, it is easy to show by (6) that $BH=GC$. Then, by (4), we have

$$\sin\angle BHF=\sin\angle CGF. \tag{7}$$

Since

$\angle BHF+\angle BFH<180° \Rightarrow \angle BHF+\angle CGF+\angle FCG<180° \Rightarrow \angle BHF+\angle CGF<180°$,

it follows from (7) that

$$\angle BHF=\angle CGF.$$

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Vectors and Analytic Geometry

Vectors are a natural way of representing line segments in space. Once thus represented, the manipulation of the vectors may proceed without much reference to the diagram, which is especially useful in three-dimensional geometry. The final two proofs involve vectors - the last proof having an analytic geometry flavour by framing the diagram within a coordinate system.

Proof 4

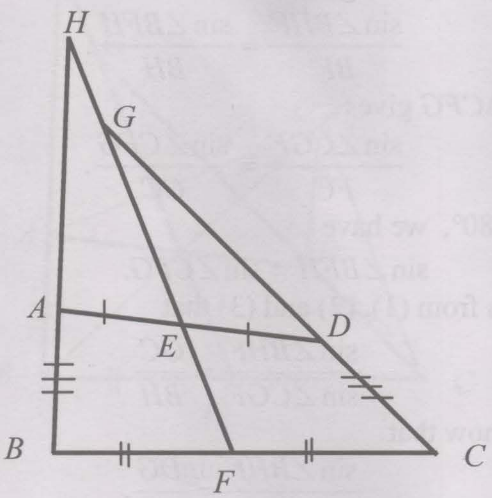

Figure 4

$\overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BF} = \frac{1}{2}\overrightarrow{DA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}.$

$\overrightarrow{EF} = \overrightarrow{ED} + \overrightarrow{DC} + \overrightarrow{CF} = \frac{1}{2}\overrightarrow{AD} + \overrightarrow{DC} + \frac{1}{2}\overrightarrow{CB}.$

Hence $\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{DC})$.

Since $(\overrightarrow{AB} + \overrightarrow{DC}) \circ (\overrightarrow{AB} - \overrightarrow{DC}) = AB^2 - DC^2 = 0$,

we have $\overrightarrow{EF} \circ (\overrightarrow{AB} - \overrightarrow{DC}) = 0$.

Hence, we have

$\overrightarrow{EF} \circ \overrightarrow{AB} = \overrightarrow{EF} \circ \overrightarrow{DC}$

$EF \times AB \times \cos\angle BHF = EF \times DC \times \cos\angle CGF$

$\cos\angle BHF = \cos\angle CGF$.

Since the two angles are between $0°$ and $180°$, we have $\angle BHF = \angle CGF$.

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Proof 5

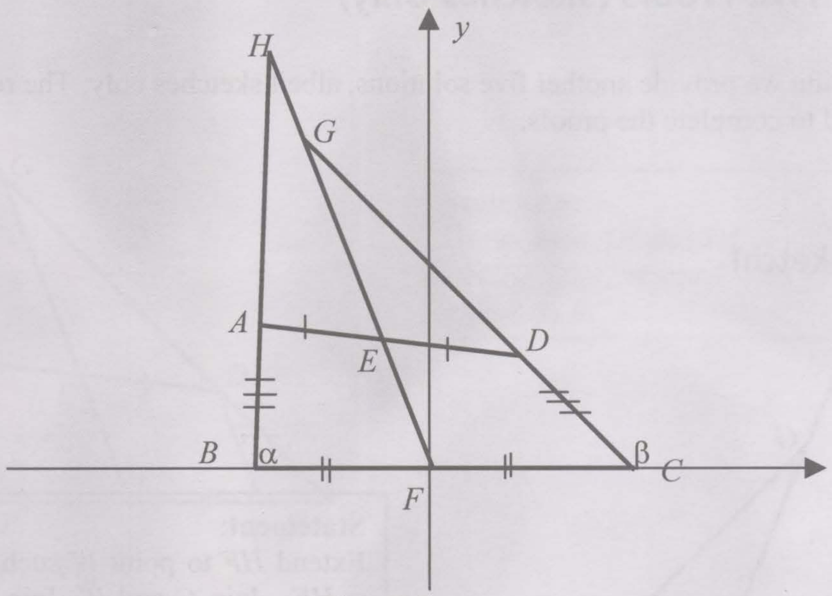

Figure 5

Let F be the origin of the coordinate system, and B and C be the points $(-a, 0)$ and $(a, 0)$ respectively. Let $CD = AB = b$. Then the coordinates of points A and D are $(-a+b\cos\alpha, b\sin\alpha)$ and $(a+b\cos\beta, b\sin\beta)$ for some α and β. Since E is the midpoint of AD, the coordinates of E are $(\frac{1}{2}b(\cos\alpha+\cos\beta), \frac{1}{2}b(\sin\alpha+\sin\beta))$.

Thus the vectors $\overrightarrow{BA}$, $\overrightarrow{FE}$ and $\overrightarrow{CD}$ are

$\overrightarrow{BA} = (b\cos\alpha, b\sin\alpha)$,

$\overrightarrow{FE} = (\frac{1}{2}b(\cos\alpha+\cos\beta), \frac{1}{2}b(\sin\alpha+\sin\beta))$,

$\overrightarrow{CD} = (b\cos\beta, b\sin\beta)$.

Now,

$\overrightarrow{BA}\circ\overrightarrow{FE} = (b\cos\alpha \times \frac{1}{2}b(\cos\alpha+\cos\beta)) + (b\sin\alpha \times \frac{1}{2}b(\sin\alpha+\sin\beta)) = \frac{1}{2}b^2(1+\cos(\alpha-\beta))$.

Similarly, $\overrightarrow{CD}\circ\overrightarrow{FE} = \frac{1}{2}b^2(1+\cos(\alpha-\beta))$.

Since $BA = CD$, $\overrightarrow{BA}\circ\overrightarrow{FE} = BA \times EF \times \cos\angle BHF$, $\overrightarrow{CD}\circ\overrightarrow{FE} = CD \times EF \times \cos\angle CGF$, we have

$$\cos\angle BHF = \cos\angle CGF.$$

Since the two angles are between 0 and 180°, we have $\angle BHF = \angle CGF$.

Note: When other approaches seem to fail, using coordinate geometry approach (as above) usually works. However, it takes quite a bit of skill to frame the diagram using a convenient coordinate system and setting the correct number of variables.

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Another Five Proofs (Sketches Only)

In this section we provide another five solutions, albeit sketches only. The reader is encouraged to complete the proofs.

Proof 6 (Sketch)

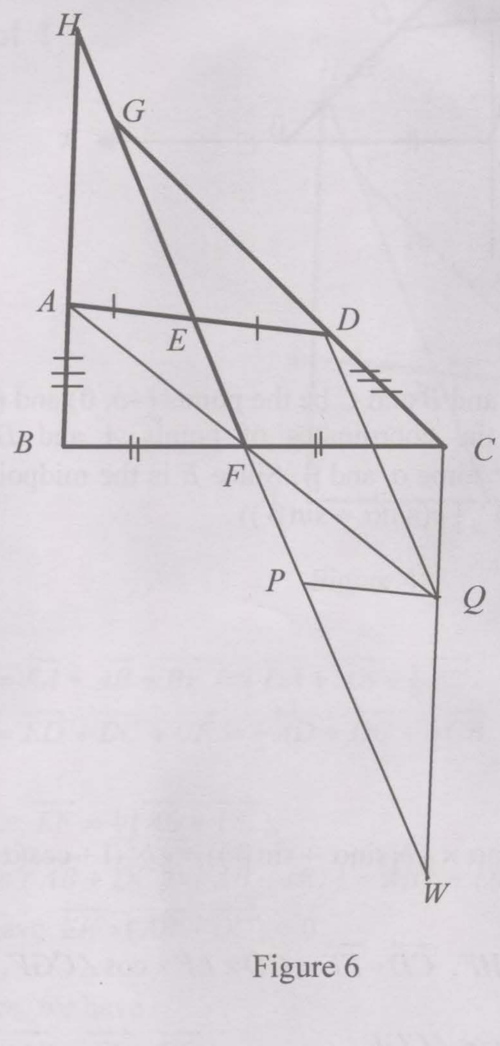

Figure 6

Statement:
Extend HF to point W such that $FW = HF$. Join C and W. Join A and F and extend AF to point Q on CW. Draw line QP parallel to DA, where P is a point on FW.

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Proof 7 (Sketch)

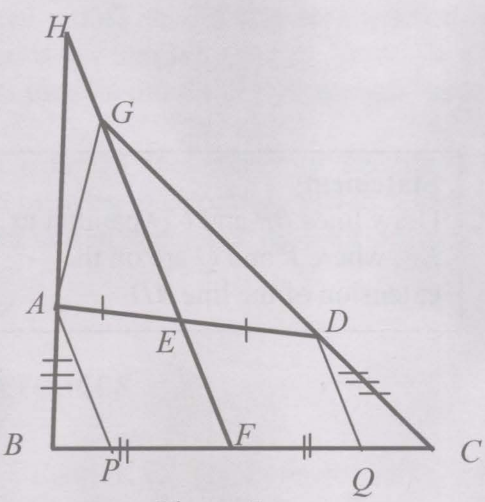

Figure 7

Statement:
Draw lines AP and DQ such that $AP // EF // DQ$, where P and Q are points on BC.

Proof 8 (Sketch)

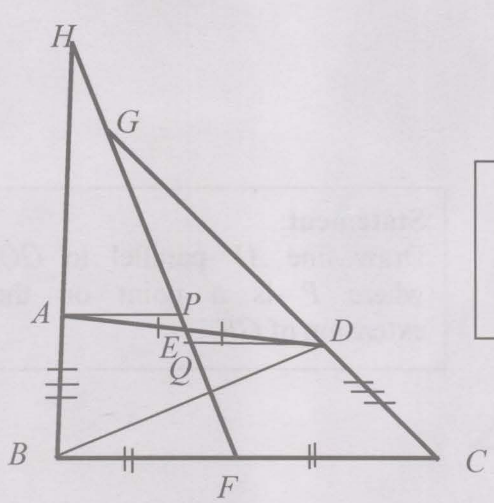

Figure 8

Statement:
Draw lines AP and DQ parallel to BC, where P and Q are two points on HF.

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Proof 9 (Sketch)

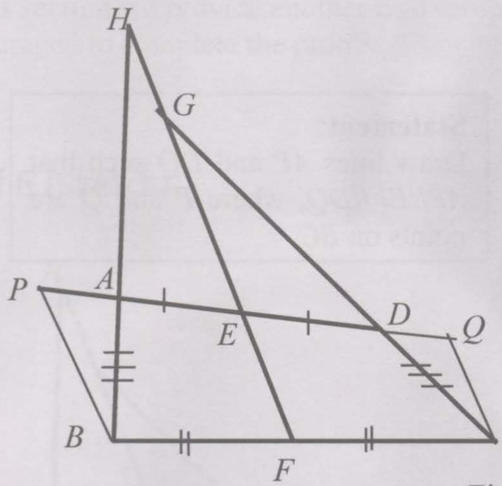

Figure 9

Statement:
Draw lines BP and CQ parallel to EF, where P and Q are on the extension of the line AD.

Proof 10 (Sketch)

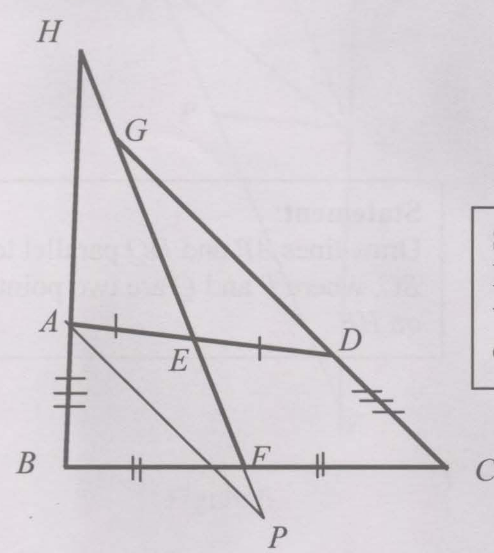

Figure 10

Statement:
Draw line AP parallel to GC, where P is a point on the extension of GF.

Conclusion

The ten proofs should convince mathematics problem solvers that there are often many solutions to a single problem. In fact, in a book by Elisha Loomis [1], there are 365 more or less distinct proofs of Pythagoras' Theorem!

We encourage the reader to attempt a nice geometry (or some other mathematics) problem using different approaches and we trust that the reader will then gain greater insight into mathematics from the approaches used.

References

1. Loomis, E., The Pythagorean Proposition, National Council of Teachers of Mathematics, (1968).